

A complete set of multidimensional Bell inequalities

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Abstract. We give a multidimensional generalisation of the complete set of Bell-correlation inequalities given by Werner and Wolf in [31], and by Żukowski and Brukner in [32], for the two-dimensional case. Our construction applies for the n parties, two-observables case, where each observable is d -valued. The d^{2n} inequalities obtained involve homogeneous polynomials. They define the facets of a polytope in a complex vector space of dimension d^n . We detail the inequalities obtained in the case $d = 3$ and, from them, we recover known inequalities. We finally explain how the violations of our inequalities by Quantum Mechanics can be computed and could be observed, when using unitary observables.

1. Introduction

The search for Bell inequalities has been the subject of a lot of work. Let us recall briefly what the matter is. Assume that a physical system is made of n subsystems. For each subsystem, a set of m different observables is considered. The outcomes of each of the nm observables belong to a set of cardinality d . The problem is to find inequalities which must be satisfied when a local realistic model is assumed.

The first such inequalities were provided by Bell [4] for the case $(n, m, d) = (2, 2, 2)$. It was also shown that Quantum Mechanics violate these inequalities. The CHSH inequalities given in [7] were shown in [9] to be a complete set for the case $(2, 2, 2)$. This means that these inequalities provide necessary and sufficient conditions for the existence of a local realistic model.

The authors of [31] and of [32] gave a complete set of 2^{2^n} Bell inequalities for dichotomic observables, with arbitrary number of parties (case $(n, 2, 2)$). The structure of these inequalities was further studied in [26], where a recursive method to compute Bell inequalities is also given. The tool for this construction was the Walsh-Hadamard transform of Boolean functions. See also [29] which gives some insight and useful details.

A method to obtain a complete set of dichotomic Bell inequalities was given in [24]. It has notably been used to exhibit a complete set for the case $(2, 3, 2)$.

The multidimensional case has also been considered in numerous references. Reasons to explore beyond the two-dimensional case include that multidimensional entangled quantum states are known to be more resistant to noise, and that they can lead to stronger violations of local realism [16]. Also there are specific uses of the tridimensional case for quantum cryptography [18]. The pioneer work for multiple outcome Bell inequalities was [8], where a family of multidimensional Bell inequalities, that generalize CHSH, was obtained. Moreover, these inequalities have been later proved tight [21].

In the literature, mostly two types of Bell inequalities are considered. Inequalities of the first type involve joint probabilities (probabilities $p(X_1 = \mu_1, \dots, X_n = \mu_n)$ that a given list of measurements outputs a given list of outcomes). Inequalities of the second type involve correlation functions. The correlation function of n discrete random variables X_1, \dots, X_n with values in a finite set \mathcal{U} is given by

$$E(X_1 \cdots X_n) = \sum_{\mu_1, \dots, \mu_n \in \mathcal{U}} \mu_1 \cdots \mu_n p(X_1 = \mu_1, \dots, X_n = \mu_n).$$

Thus, it is possible to convert any Bell inequality (obtained for a local realistic model) of the second type to one of the first type. The converse is also true when dealing with dichotomic observables, as shown for example in [19]. For a higher number d of outcomes, this equivalence does not hold. Inequalities for joint probabilities and higher d have been given for example in [8]. Inequalities with correlations functions and

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2. Multidimensional discrete Fourier transform

$d = 3$ have yet been considered in [11], [18], [17], [30]. Our work is about inequalities involving correlations functions.

We use a geometrical approach. Froissart [10] has apparently been the first to do so, and then the authors of [13] independently. It was shown in [23] that the local-realistic domain, for joint probabilities, is a convex polytope. When considering correlation functions inequalities, the geometrical approach can be real or complex, and these two approaches give different local realistic domains (for multiple outcomes). For the real approach, it was shown in [20], that even in the case of multiple outcomes, but all belonging to the interval $[-1, 1]$, the local realistic domain is the same as the one given in [31] and [32].

We use instead a complex approach, and map the outcomes to the set of d -th roots of unity in \mathbb{C} , as in [11], [18], [17], [30]. In this setting, no complete set has been given yet, beyond the two-dimensional case. We explicit here a polytope corresponding to the local realistic domain, when $d > 2$. This polytope belongs to a complex vector space of dimension d^n .

Our inequalities are tight. This means that they define the facets of the polytope. The problem of obtaining all the (tight) inequalities was only solved for outcomes belonging to $[-1, 1]$ (in [9], [24] with joint probabilities, [31] and [32] with correlation functions).

Our inequalities involve products and powers of observables, arranged in homogeneous polynomial expressions. Powers of observables have already been used in [30]. It turns out that the method developed for $(n, 2, 2)$ generalizes pretty well for the multidimensional, two-observables per party case, by means of multidimensional discrete Fourier transform. With this tool, we are able to give a complete set of tight Bell inequalities for the case $(n, 2, d)$.

In this paper, we first presents background about multidimensional Fourier transform (DFT for short). Then we recall some facts about the duality of polytopes in (finite dimensional) Hilbert spaces and study some useful relations between DFT and duality. Then we produce d^{d^n} Bell inequalities, involving correlation functions, which generalize those obtained in [31]. We study the polynomials involved in these inequalities and give some facts about the symmetries observed. Then we prove that our Bell inequalities form a complete set of tight ones. We illustrate our results in the case $d = 3$ and then explains the connection between our inequalities and known inequalities (the CHSH inequality for qutrits and the CGLMP inequality for $d = 3$). Finally, in the last section, we explain how violations of our Bell inequalities by Quantum Mechanics can be computed and observed, with the use of unitary observables.

2. Multidimensional discrete Fourier transform

There are numerous references for the discrete Fourier transform. One of them is [5]. However, we give here all the material we need for our purposes.

Maps from \mathbb{Z}_d^n to the set of d -th roots of 1

The main tool for the classification of dichotomic Bell inequalities is the Walsh-Hadamard transform for Boolean functions. For our generalisation of the dichotomic case, we will use d -valued functions and multidimensional discrete Fourier transform.

There are two equivalent ways to define Boolean functions: it can be a map F from $\{0, 1\}^n$ to $\{0, 1\}$ (additive convention), or a map f from $\{0, 1\}^n$ to $\{1, -1\}$ (multiplicative convention). The equivalence is of course given by $f = (-1)^F$. The multiplicative convention is more comfortable when dealing with Walsh-Hadamard transforms. We also adopt a multiplicative convention, and the considered functions will take their values in the set

$$\mathcal{U} = \{1, \omega, \dots, \omega^{d-1}\} \quad \text{where } \omega = \exp(2i\pi/d). \quad (1)$$

We put $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$ and denote by \mathbb{Z}_d^n the set of n -tuples with components in \mathbb{Z}_d ($d, n \in \mathbb{N}^*$). Also, we denote by \mathcal{F} or $\mathcal{F}_{d,n}$ the set of maps from \mathbb{Z}_d^n to \mathcal{U} . There are d^{d^n} such functions.

2. Multidimensional discrete Fourier transform

The DFT

Let f be a map from \mathbb{Z}_d^n to the complex field \mathbb{C} (or to \mathcal{U} as a particular case). The (multidimensional) discrete Fourier transform of f is the map $\text{DFT } f = \hat{f}$, also from \mathbb{Z}_d^n to \mathbb{C} , defined by

$$\hat{f}(r_1, \dots, r_n) = \sum_{s_1, \dots, s_n \in \mathbb{Z}_d} \omega^{r_1 s_1 + \dots + r_n s_n} f(s_1, \dots, s_n) \quad (2)$$

or, written in compact form, $\hat{f}(r) = \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot s} f(s)$ where $r \cdot s = \sum_{i=1}^n r_i s_i$ is the standard scalar product of the n -tuples r and s .

We denote as H_d the matrix $(\omega^{ij})_{0 \leq i, j \leq d-1}$. The n -th tensor power of H_d is the $D \times D$ matrix, with $D = d^n$, given by

$$H_d^{\otimes n} := (\omega^{r \cdot s})_{r, s \in \mathbb{Z}_d^n}.$$

The matrices $H_d^{\otimes n}$ can be built up from blocks using recursion on n :

$$H_d^{\otimes 0} = (1) \quad \text{and} \quad H_d^{\otimes n} = (\omega^{ij} H_d^{\otimes n-1})_{0 \leq i, j \leq d-1}. \quad (3)$$

These matrices are a generalization of the usual Hadamard matrices which are obtained in the special case $d = 2$ (hence $\omega = -1$).

A map f from \mathbb{Z}_d^n to \mathbb{C} can be identified to the vector of its values $(f(s))_{s \in \mathbb{Z}_d^n}$. The (column) vector of the values of \hat{f} can be obtained applying the matrix $H_d^{\otimes n}$ to the (column) vector of the values of f :

$$\begin{pmatrix} \hat{f}(0, \dots, 0, 0) \\ \hat{f}(0, \dots, 0, 1) \\ \vdots \\ \hat{f}(d-1, \dots, d-1) \end{pmatrix} = H_d^{\otimes n} \begin{pmatrix} f(0, \dots, 0, 0) \\ f(0, \dots, 0, 1) \\ \vdots \\ f(d-1, \dots, d-1) \end{pmatrix}.$$

Hence, the map $\text{DFT}: f \mapsto \hat{f}$ is a linear map from \mathbb{C}^{d^n} to itself.

Inverse DFT

Let also define $H_d^{*\otimes n}$ the matrix $(\omega^{-r \cdot s})_{r, s \in \mathbb{Z}_d^n}$. It can be checked that

$$H_d^{*\otimes n} H_d^{\otimes n} = d^n I.$$

Hence, the inverse transform DFT^{-1} is obtained by

$$f(s_1, \dots, s_n) = \frac{1}{d^n} \sum_{r_1, \dots, r_n \in \mathbb{Z}_d} \omega^{-(r_1 s_1 + \dots + r_n s_n)} \hat{f}(r_1, \dots, r_n)$$

or, in compact form, $f(s) = \frac{1}{d^n} \sum_{r \in \mathbb{Z}_d^n} \omega^{-r \cdot s} \hat{f}(r)$.

In the particular case $d = 2$, the multidimensional discrete Fourier transform is the Walsh-Hadamard transform of Boolean functions:

$$\hat{f}(w) = \sum_{x \in \{0,1\}^n} (-1)^{w \cdot x} f(x)$$

(using the multiplicative convention: $f(x) \in \{1, -1\}$).

3. Convex hulls

Some easy results

Some easy results can be derived from the definition given by Equation (2), between the discrete Fourier transforms of two elements of $\mathcal{F}_{d,n}$ which are related in some way:

2.1. — Proposition. Put $\hat{f} = \text{DFT } f$ and $\hat{g} = \text{DFT } g$ where f and g belong to $\mathcal{F}_{d,n}$.

- (a) If $g(s) = f(-s)$ for all $s \in \mathbb{Z}_d^n$, then $\hat{g}(r) = \hat{f}(-r)$ for all $r \in \mathbb{Z}_d^n$.
- (b) If $g(s) = f(-s)^*$ for all $s \in \mathbb{Z}_d^n$, then $\hat{g}(r) = \hat{f}(r)^*$ for all $r \in \mathbb{Z}_d^n$ (* denotes complex conjugation).
- (c) Let $\delta \in \mathbb{Z}_d^n$. If $g(s) = f(s + \delta)$ for all $s \in \mathbb{Z}_d^n$ (addition in \mathbb{Z}_d^n is assumed component-wise and modulo d), then $\hat{g}(r) = \omega^{-r \cdot \delta} \hat{f}(r)$ for all $r \in \mathbb{Z}_d^n$.
- (d) Let $\delta \in \mathbb{Z}_d^n$. If $g(s) = \omega^{\delta \cdot s} f(s)$ for all $s \in \mathbb{Z}_d^n$, then $\hat{g}(r) = \hat{f}(r + \delta)$ for all $r \in \mathbb{Z}_d^n$.
- (e) Let σ be a permutation of the set $\{1, \dots, n\}$. For $s = (s_1, \dots, s_n) \in \mathbb{Z}_d^n$, we use the shorthand notation $\sigma(s) = (s_{\sigma(1)}, \dots, s_{\sigma(n)})$. If $g(s) = f(\sigma(s))$ for all $s \in \mathbb{Z}_d^n$, then $\hat{g}(r) = \hat{f}(\sigma(r))$ for all $r \in \mathbb{Z}_d^n$.

PROOF — We show only the last two assertions and leave the first three to the reader. Assume that $g(s) = \omega^{\delta \cdot s} f(s)$ for all $s \in \mathbb{Z}_d^n$. Then

$$\hat{g}(r) = \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot s} g(s) = \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot s} \omega^{\delta \cdot s} f(s)$$

for all $r \in \mathbb{Z}_d^n$. Hence,

$$\hat{g}(r - \delta) = \sum_{s \in \mathbb{Z}_d^n} \omega^{(r - \delta) \cdot s} \omega^{\delta \cdot s} f(s) = \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot s} f(s) = \hat{f}(r).$$

This proves assertion (d). Assume now that $g(s) = f(\sigma(s))$ for all $s \in \mathbb{Z}_d^n$. Then, for all $r \in \mathbb{Z}_d^n$,

$$\begin{aligned} \hat{g}(r) &= \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot s} g(s) = \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot s} f(\sigma(s)) \\ &= \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot \sigma^{-1}(s)} f(s) \quad \text{because } \sigma \text{ induces a permutation on } \mathbb{Z}_d^n. \end{aligned}$$

Hence

$$\hat{g}(\sigma^{-1}(r)) = \sum_{s \in \mathbb{Z}_d^n} \omega^{\sigma^{-1}(r) \cdot \sigma^{-1}(s)} f(s) = \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot s} f(s) = \hat{f}(r).$$

This proves assertion (e). □

3. Convex hulls

Let $D \in \mathbb{N}$. We denote $\langle \beta, \gamma \rangle = \sum_{i=1}^D \beta_i^* \gamma_i$ the usual Hermitian inner product in \mathbb{C}^D . The complex vector space \mathbb{C}^D can also be viewed as a vector space over \mathbb{R} , with dimension $2D$. Each element $\beta \in \mathbb{C}^D$ can be alternatively written as a D -uple $(\beta_1, \dots, \beta_D)$ of coordinates belonging to \mathbb{C} or as a $2D$ -uple $(x_1, y_1, \dots, x_D, y_D)$ of coordinates belonging to \mathbb{R} , with the relations $\beta_k = x_k + iy_k$. Recall that the real part of the inner product $\langle \cdot, \cdot \rangle$ is nothing more than the usual scalar product in \mathbb{R}^{2D} :

$$\text{Re} \langle \beta, \gamma \rangle = \text{Re} \sum_{k=1}^D \beta_k^* \gamma_k = \sum_{k=1}^D (x_k z_k + y_k t_k) \quad \begin{array}{l} \text{if } \beta_k = x_k + iy_k \text{ and } \gamma_k = z_k + it_k, \\ \text{with } x_k, y_k, z_k, t_k \in \mathbb{R}. \end{array}$$

Let S be a subset of \mathbb{C}^D . The *convex hull* of S is the set

$$\text{Hull } S := \left\{ \sum_k p_k \beta_k \text{ with } \beta_k \in S \text{ and } p_k \in \mathbb{R}_+ \text{ such that } \sum_k p_k = 1 \right\}.$$

The *dual* (or *polar*) of the set S is, by definition, the set

$$T = S^\circ := \{ \gamma \in \mathbb{C}^D \mid \text{Re} \langle \beta, \gamma \rangle \leq 1, \forall \beta \in S \}. \quad (4)$$

When S is a polytope containing 0, the vertices of the dual T correspond to the facets of S . To be precise, γ is a vertex of T if and only if the hyperplane defined by the equation $\text{Re} \langle \beta, \gamma \rangle = 1$ contains a facet of S .

The following result holds (the bipolar Theorem, see [27]):

3. Convex hulls

3.1. — Theorem. For any subset S of \mathbb{C}^D containing 0 , the dual $S^{\circ\circ}$ of the dual of S is the convex hull of S .

The hull of \mathcal{U} and its dual

We assume here $d > 2$. The convex hull of the set \mathcal{U} is a regular polygon. The dual of \mathcal{U} is also a regular polygon with d vertices (see Figure 1):

3.2. — Lemma. The dual \mathcal{U}° of \mathcal{U} (with $d > 2$) is the polygon with vertices set:

$$\mathcal{V} = \left\{ \frac{1}{\cos(\pi/d)} \exp\left(\frac{2k+1}{d}i\pi\right) \mid k = 0, \dots, d-1 \right\}.$$

PROOF — For $\beta_k = \exp\left(\frac{2ki\pi}{d}\right) \in \mathcal{U}$ and $\gamma_l = \exp\left(\frac{(2l+1)i\pi}{d}\right) / \cos\left(\frac{\pi}{d}\right) \in \mathcal{V}$ we have

$$\operatorname{Re}\langle \beta_k, \gamma_l \rangle = \frac{\operatorname{Re}\left(\exp(-2ki\pi/d) \exp\left(\frac{(2l+1)i\pi}{d}\right)\right)}{\cos(\pi/d)} = \frac{\cos\left(\frac{(2l+1-2k)\pi}{d}\right)}{\cos(\pi/d)}.$$

Thus, $\operatorname{Re}\langle \beta_k, \gamma_l \rangle = 1$ when $k = l$ or when $k = l + 1$ (the vertex γ_l of \mathcal{U}° corresponds to the edge $\delta_l = (\beta_l, \beta_{l+1})$ of $\operatorname{Hull} \mathcal{U}$). For the other values of k , we have $\operatorname{Re}\langle \beta_k, \gamma_l \rangle < 1$ because β_k is in the half-plane delimited by δ_l and containing 0 . \square

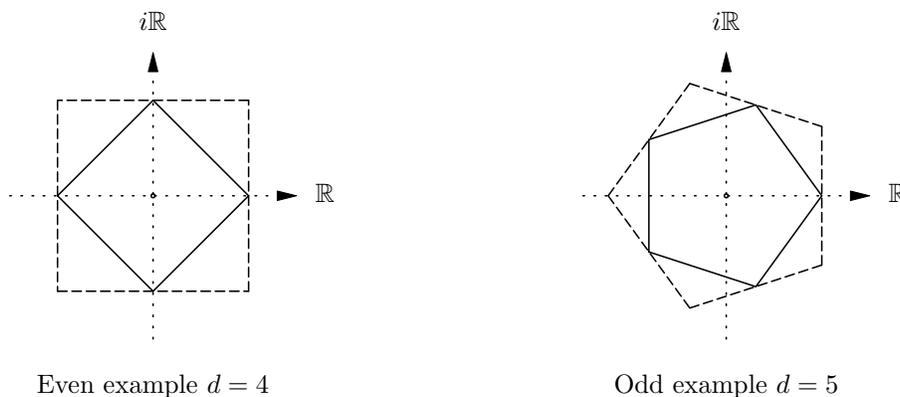


Figure 1. The boundaries of the convex hull of \mathcal{U} (solid) and its dual (dashed)

3.3. — Lemma. Define $\rho = \exp(i\pi/d)$. For each $\beta \in \operatorname{Hull} \mathcal{U}$, the following inequality holds:

$$\operatorname{Re}(\rho\beta) \leq \cos(\pi/d).$$

PROOF — From Lemma 3.2, we have $\mathcal{U}^\circ = \frac{\rho}{\cos(\pi/d)} \operatorname{Hull} \mathcal{U}$. Hence $\rho\beta = \cos(\pi/d)\gamma$ for some $\gamma \in \mathcal{U}^\circ$. Thus, $\operatorname{Re}(\rho\beta) = \cos(\pi/d) \operatorname{Re}(\gamma)$. But we have $\operatorname{Re}(\gamma) = \operatorname{Re}\langle 1, \gamma \rangle \leq 1$ because $1 \in \mathcal{U}$. The result follows because $\cos(\pi/d) > 0$ (we assumed $d > 2$, note also that case $d = 2$ is trivially true). \square

4. Homogeneous Bell inequalities

Duality and DFT

As in Section 2, we put $D = d^n$. The map DFT is linear and its matrix $U = H_d^{\circ n}$ (in the canonical basis of \mathbb{C}^D) satisfies $U^\dagger U = DI$, where U^\dagger is the conjugate transpose of U . This has some useful consequences.

3.4. — Lemma. *Assume that $\beta, \gamma \in \mathbb{C}^D$, and put $\hat{\beta} = \text{DFT } \beta$ and $\hat{\gamma} = \text{DFT } \gamma$. We have $\langle \hat{\beta}, \hat{\gamma} \rangle = D \langle \beta, \gamma \rangle$.*

PROOF — If we identify β and γ with the column vectors of their coordinates in the canonical basis we can write:

$$\langle \hat{\beta}, \hat{\gamma} \rangle = \hat{\beta}^\dagger \hat{\gamma} = (U\beta)^\dagger U\gamma = \beta^\dagger U^\dagger U\gamma = D\beta^\dagger \gamma = D \langle \beta, \gamma \rangle$$

as claimed. □

3.5. — Proposition. *Let Γ be a polytope in \mathbb{C}^D containing 0, and denote by $\hat{\Gamma}$ its image (which is also a polytope, by linearity of DFT) under the map DFT. We have the following relations between their duals:*

$$\widehat{\hat{\Gamma}} = D\hat{\Gamma}^\circ.$$

PROOF — For $\beta \in \mathbb{C}^D$, we have $\beta \in \Gamma^\circ$ if and only if $\langle \beta, \gamma \rangle \leq 1$ for all $\gamma \in \Gamma$. From Lemma 3.4, this is equivalent to $\langle \hat{\beta}, \hat{\gamma} \rangle \leq D$ for all $\gamma \in \Gamma$. This condition can be written $\langle \frac{1}{D}\hat{\beta}, \hat{\gamma} \rangle \leq 1$, or therefore $\frac{1}{D}\hat{\beta} \in \hat{\Gamma}^\circ$. Finally, it is equivalent to $\hat{\beta} \in D\hat{\Gamma}^\circ$. □

4. Homogeneous Bell inequalities

Let n be the number of parties. For each party, we consider two observables, denoted by \hat{A}_i and \hat{B}_i (for $1 \leq i \leq n$). The outcomes of each measure are assumed to belong to the set \mathcal{U} defined in (1), with $d \geq 2$.

Recall, from the identity

$$1 - X^d = (1 - X)(1 + X + X^2 + \dots + X^{d-1}),$$

that the roots of the polynomial $1 + X + \dots + X^{d-1}$ are the elements of $\mathcal{U} \setminus \{1\}$. Recall also that $\sum_{u \in \mathcal{U}} u^k$ evaluates to d when k is a multiple of d but is zero otherwise. If $a_i, b_i \in \mathcal{U}$, there exists an integer $r_i \in \mathbb{Z}_d$ such that $a_i/b_i = \omega^{r_i}$. Let also $s_i \in \mathbb{Z}_d$. Then

$$\begin{aligned} a_i^{d-1} + \omega^{s_i} a_i^{d-2} b_i + \dots + \omega^{(d-1)s_i} b_i^{d-1} &= a_i^{d-1} (1 + \omega^{s_i - r_i} + \dots + \omega^{(d-1)(s_i - r_i)}) \\ &= \begin{cases} a_i^{d-1} d & \text{if } r_i = s_i, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let now f be any map from \mathbb{Z}_d^n to \mathcal{U} . We have

$$\sum_{s \in \mathbb{Z}_d^n} f(s) \prod_{i=1}^n (a_i^{d-1} + \omega^{s_i} a_i^{d-2} b_i + \dots + \omega^{r_i s_i} a_i^{d-1-r_i} b_i^{r_i} + \dots + \omega^{(d-1)s_i} b_i^{d-1}) = u d^n \quad (5)$$

where $u \in \mathcal{U}$, because in this sum, exactly one term is non-zero (the one corresponding to $s_i = r_i$ for each i).

If we expand the products in (5), we get

$$\begin{aligned} u d^n &= \sum_{s \in \mathbb{Z}_d^n} f(s) \sum_{r \in \mathbb{Z}_d^n} \prod_{i=1}^n \omega^{s_i r_i} a_i^{d-1-r_i} b_i^{r_i} \\ &= \sum_{s \in \mathbb{Z}_d^n} f(s) \sum_{r \in \mathbb{Z}_d^n} \omega^{r \cdot s} a^r \quad \text{where } a^r := \prod_{i=1}^n a_i^{d-1-r_i} b_i^{r_i} \\ &= \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) a^r \quad \text{where } \hat{f} = \text{DFT } f. \end{aligned}$$

5. Homogeneous Bell polynomials

Now, if the a_i and b_i are random variables we can write, about expected values:

$$\sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) E(a^r) \in d^n \text{ Hull } \mathcal{U}.$$

From Lemma 3.3, we obtain:

$$\text{Re} \left(\rho \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) E(a^r) \right) \leq d^n \cos(\pi/d).$$

When $d > 2$, this also can be written:

$$\text{Re} \left(\frac{\rho}{d^n \cos(\pi/d)} \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) E(a^r) \right) \leq 1 \quad \text{for } f \in \mathcal{F}_{d,n}. \quad (6)$$

We call these relations homogeneous Bell inequalities. There are d^D of them.

5. Homogeneous Bell polynomials

We now study the polynomials in $2n$ variables A_i and B_i (for $1 \leq i \leq n$) which are involved in the homogeneous Bell inequalities. Some Bell polynomials were defined in [31] for $d = 2$. As a generalisation to the multidimensional case, we define the homogeneous Bell polynomials to be

$$\mathcal{P}_f = \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) A^r \quad \text{where } A^r := \prod_{i=1}^n A_i^{d-1-r_i} B_i^{r_i} \quad (7)$$

where f is any map from \mathbb{Z}_d^n to \mathcal{U} . Let us denote $\mathcal{H}_{d,n}$ the set of these polynomials. Each element of $\mathcal{H}_{d,n}$ is a homogeneous polynomial of degree $n(d-1)$. Note that in view of Section 9, we consider \mathcal{P}_f as a *non commutative* polynomial. More precisely, each A_i is **not** assumed to commute with B_i , while A_i and B_i do commute with A_j and B_j for $i \neq j$.

As in [27] where the case $d = 2$ is handled, we give a recursive construction of the homogeneous Bell polynomials. This construction is a direct consequence of Equation (3). If $\mathcal{P}_0, \dots, \mathcal{P}_{d-1}$ are homogeneous Bell polynomials in the $2(n-1)$ the variables A_i, B_i with $1 \leq i \leq n-1$, then we get a homogeneous Bell polynomial in $2n$ variables by the d -ary operation \bowtie :

$$\mathcal{P}_0 \bowtie \dots \bowtie \mathcal{P}_{d-1} := \sum_{r_n=0}^{d-1} \left(\sum_{t=0}^{d-1} \omega^{r_n t} \mathcal{P}_t \right) A_n^{d-1-r_n} B_n^{r_n}.$$

Conversely, every element of the set $\mathcal{H}_{d,n}$ can be obtained this way.

For example, with $d = 2$, the polynomials obtained are ± 1 for $n = 0$, $\pm 2A_1$ and $\pm 2B_1$ for $n = 1$, and

$$\begin{aligned} \pm 4A_1A_2, & \quad \pm 2(-A_1A_2 + A_1B_2 + B_1A_2 + B_1B_2), \\ \pm 4A_1B_2, & \quad \pm 2(A_1A_2 - A_1B_2 + B_1A_2 + B_1B_2), \\ \pm 4B_1A_2, & \quad \pm 2(A_1A_2 + A_1B_2 - B_1A_2 + B_1B_2), \\ \pm 4B_1B_2, & \quad \pm 2(A_1A_2 + A_1B_2 + B_1A_2 - B_1B_2), \end{aligned}$$

for $n = 2$ (we recognize the polynomials involved in the CHSH inequalities). Examples for $d = 3$ will be given in Section 7.

6. The classical domain

Symmetries

The set $\mathcal{H}_{d,n}$ of homogeneous Bell polynomials has some symmetries we briefly discuss now. They are consequences of Proposition 2.1.

a. If the maps f and $g \in \mathbb{Z}_d^n$ are the same, up to the order of their arguments:

$$g(s_1, \dots, s_n) = f(s_{\sigma(1)}, \dots, s_{\sigma(n)}) \quad \text{for all } s \in \mathbb{Z}_d^n,$$

for some permutation σ , then the polynomial \mathcal{P}_g can be obtained from \mathcal{P}_f by changing each variable A_i (resp. B_i) to $A_{\sigma(i)}$ (resp. $B_{\sigma(i)}$). This symmetry corresponds to the fact that the n subsystems are indistinguishable.

b. If, for some i_0 ,

$$g(s) = \omega^{-s_{i_0}} f(s) \quad \text{for all } s \in \mathbb{Z}_d^n,$$

then, from Proposition 2.1, we have $\hat{g}(r) = \hat{f}(r - \delta)$ for all $r \in \mathbb{Z}_d^n$, where $\delta = (0, \dots, 0, 1, 0, \dots, 0)$ has its only non-null component at index i_0 . Hence, we obtain

$$\mathcal{P}_g = \sum_{r \in \mathbb{Z}_d^n} \hat{g}(r) A^r = \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r - \delta) A^r = \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) A^{r+\delta}.$$

This shows that we obtain \mathcal{P}_g from \mathcal{P}_f by the circular monomial substitution

$$A_{i_0}^{d-1} \longrightarrow A_{i_0}^{d-2} B_{i_0} \longrightarrow \dots \longrightarrow A_{i_0} B_{i_0}^{d-2} \longrightarrow B_{i_0}^{d-1} \longrightarrow A_{i_0}^{d-1}.$$

Also, the set $\mathcal{H}_{d,n}$ is invariant, under the swap operation $A_{i_0} \leftrightarrow B_{i_0}$ (this can be algebraically checked with the help of Proposition 2.1(a)). Hence, for each i_0 , the set $\mathcal{H}_{d,n}$ is invariant under the action of the dihedral group of order $2d$ over the monomials made of the variables A_{i_0} and B_{i_0} .

c. Of course, the set $\mathcal{H}_{d,n}$ is also invariant under multiplication by ω , and by complex conjugation (Proposition 2.1(b) can be used to check this latter fact).

6. The classical domain

We now show that the homogeneous Bell inequalities obtained in Section 4 are tight and completely characterize a local realistic model, for $n \in \mathbb{N}^*$ parties, $m = 2$ observables for each site, and d -outcomes measurements with $d > 2$.

The values a_i and b_i , when a local realistic model is applied, of these $2n$ observables are assumed to belong to the set \mathcal{U} . We consider the monomials

$$A^s = \prod_{i=1}^n A_i^{d-1-s_i} B_i^{s_i}$$

which appear in homogeneous Bell polynomials. There are $D = d^n$ of them. For each experiment, the data set of the values obtained for these monomials form a vector $\xi = (a^s)_{s \in \mathbb{Z}_d^n}$ in \mathbb{C}^D . Our aim is to show that the domain accessible to the expected values of ξ is the polytope defined by the inequalities (6).

The polytope Ω

Put

$$\xi_r = (\omega^{r \cdot s})_{s \in \mathbb{Z}_d^n} \in \mathbb{C}^D \quad \text{for each } r \in \mathbb{Z}_d^n.$$

The $d^{n+1} = dD$ vectors $u\xi_r$, for $u \in \mathcal{U}$ and $r \in \mathbb{Z}_d^n$ are all distinct. In a local realistic model, each experimental data set assigns a value

$$\prod_{i=1}^n a_i^{d-1-s_i} b_i^{s_i} = \prod_{i=1}^n a_i^{d-1} \prod_{i=1}^n \omega^{r_i s_i} = \prod_{i=1}^n a_i^{d-1} \omega^{r \cdot s}$$

6. The classical domain

to each monomial A^s where $\omega^{r_i} = b_i/a_i$ (for $1 \leq i \leq n$). Thus, the vector ξ obtained from experimental data is one of the vectors $u\xi_r$, where $u = \prod_{i=1}^n a_i^{d-1}$, and $r = (r_i)_{1 \leq i \leq n}$ with the r_i just defined.

Conversely, it is possible to design classical experiments which assign independently any value in \mathcal{U} to the $2n$ variables and which assign any $u\xi_r$ to the data set vector ξ . Then, if the values assigned to the variables follow some probability distributions, expected values for the vectors ξ obtained, are convex combinations of the $u\xi_r$. Hence the classically accessible region for ξ is the convex hull of the $u\xi_r$, which will be denoted by Ω as it was in [31] for the case $d = 2$. The domain Ω is a polytope in \mathbb{C}^D and has dD vertices. Notice that Ω has a d -order symmetry: $\omega\Omega = \Omega$.

The polytope $\Pi = \text{DFT}^{-1} \Omega$

We can find all the inequalities defining the facets of the polytope Ω . They will be the d^D homogeneous Bell inequalities (6) we obtained in Section 4.

Let $(\pi_s)_{s \in \mathbb{Z}_d^n}$ be the canonical basis of the complex vector space \mathbb{C}^D . The discrete Fourier transform maps the π_s to the ξ_s . We consider the following polytope:

$$\Pi := \text{Hull}\{u\pi_s \mid u \in \mathcal{U}, s \in \mathbb{Z}_d^n\}.$$

Then Ω is $\hat{\Pi}$, the image of Π under DFT. To find the facets of Ω , we have to study its dual. But from Proposition 3.5,

$$\Omega^\circ = \hat{\Pi}^\circ = \frac{1}{d^n} \widehat{\Pi^\circ}. \quad (8)$$

Let's first study Π° .

6.1. — Proposition. *The vertices of the polytope Π° are the $\beta = (\beta_1, \dots, \beta_s)$ such that*

$$\beta_s = \frac{\rho}{\cos(\pi/d)} f(s) \quad \text{where } f \text{ is any element of } \mathcal{F}_{d,n}.$$

PROOF — By definition,

$$\Pi^\circ = \{\beta \in \mathbb{C}^D \mid \text{Re}\langle \beta, u\pi_s \rangle \leq 1, \forall u \in \mathcal{U}, s \in \mathbb{Z}_d^n\}.$$

Using the d -order symmetry of \mathcal{U}° , and using $\langle \beta, u\pi_s \rangle = u\langle \beta, \pi_s \rangle$, we can write

$$\Pi^\circ = \{\beta \in \mathbb{C}^D \mid \langle \beta, \pi_s \rangle \in \mathcal{U}^\circ, \forall s \in \mathbb{Z}_d^n\}.$$

We are interested with the extremal points of Π° . These are obtained when $\langle \beta, \pi_s \rangle$ are in a corner of \mathcal{U}° (see Lemma 3.2):

$$\langle \beta, \pi_s \rangle \in \mathcal{V} = \frac{\rho}{\cos(\pi/d)} \mathcal{U}.$$

Hence, there exists $f \in \mathcal{F}_{d,n}$ such that:

$$\beta_s^* = \langle \beta, \pi_s \rangle = \frac{\rho}{\cos(\pi/d)} f(s) \quad \text{for all } s \in \mathbb{Z}_d^n.$$

But Π° is symmetric under complex conjugation. Hence we can change β_s^* for β_s . □

The dual of Ω

6.2. — Theorem. *The vertices of the polytope Ω° are given by*

$$\frac{\rho}{d^n \cos(\pi/d)} (\hat{f}(r))_{r \in \mathbb{Z}_d^n} \quad \text{for } f \in \mathcal{F}_{d,n}.$$

PROOF — The result follows from Equation (8) and Proposition 6.1. □

To end this section, note that the inequalities (6) can be written

$$\text{Re}\langle \beta_f, \xi \rangle \leq 1 \quad \text{with} \quad \beta_f^* = \frac{\rho}{d^n \cos(\pi/d)} (\hat{f}(r))_{r \in \mathbb{Z}_d^n} \quad \text{and} \quad \xi = (E(a^r))_{r \in \mathbb{Z}_d^n}.$$

Hence the theorem just obtained shows that our homogeneous Bell inequalities define the facets of the polytope Ω . Thus they form a complete set of tight Bell inequalities.

7. The case $d = 3$

We illustrate our results with the first multidimensional case: $d = 3$ (sometimes called trichotomic). Note that the factor $1/\cos(\pi/d)$ in Equation (6) is maximal in this case, and this might lead to higher violations.

DFT

Here, $\omega = \exp(2i\pi/3)$ and

$$H_3^{\otimes 1} = H_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad H_3^{\otimes 2} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ 1 & 1 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega & 1 & \omega^2 \end{pmatrix}.$$

The hull of \mathcal{U} and its dual

The hull of \mathcal{U} is the triangle with vertices $1, \omega, \omega^2$ and its edges are defined by the three inequalities

$$x + \sqrt{3}y \leq 1, \quad -2x \leq 1, \quad x - \sqrt{3}y \leq 1.$$

Hence, the dual \mathcal{U}° has vertices $1 + i\sqrt{3}, -2, 1 - i\sqrt{3}$, which are obtained from the vertices of Hull \mathcal{U} by multiplication by $\exp(i\pi/3)/\cos(\pi/3) = -2\omega^2$.

Bell polynomials

We did some computations, with the help of the Magma computer algebra system. For the (virtual) case $n = 0$, the trichotomic Bell polynomials are the constant polynomials $1, \omega$ and ω^2 . For $n = 1$, there are yet 27 homogeneous trichotomic Bell polynomials. Instead of listing them all, we give for them the following compact expression:

$$u(3M + (v - 1)(A^2 + AB + B^2)) \tag{9}$$

where $u, v \in \{1, \omega, \omega^2\}$ and $M \in \{A^2, AB, B^2\}$.

For $n = 2$, there are 19683 homogeneous trichotomic Bell polynomials. Among them, 18792 are irreducible polynomials. The number of elements in $\mathcal{H}_{3,2}$ with only real coefficients is 81 (the aim of this criterion here is just to reduce the list size). We can list them, up to the symmetries discussed in Section 5, as there remain only 4 ones:

$$\begin{aligned} & 9A_1^2A_2^2 \\ & 3(A_1^2A_2^2 - A_1^2B_2^2 + 2A_1B_1A_2B_2 + A_1B_1B_2^2 - B_1^2A_2^2 + B_1^2A_2B_2) \\ & 3(-A_1^2A_2B_2 + A_1^2B_2^2 + A_1B_1A_2^2 + 2A_1B_1B_2^2 - B_1^2A_2^2 + B_1^2A_2B_2) \\ & 3(2A_1^2A_2^2 - A_1^2A_2B_2 - A_1^2B_2^2 + A_1B_1A_2^2 + A_1B_1A_2B_2 + A_1B_1B_2^2). \end{aligned}$$

We found also that there are 243 elements in $\mathcal{H}_{3,2}$ up to these symmetries.

Bell inequalities

The factor $\frac{\rho}{d^n \cos(\pi/d)}$ which appear in Inequalities (6) is in this case $-2\omega^2/3^n$. By changing f to ωf , we can remove the ω^2 to obtain the following homogeneous trichotomic Bell inequalities:

$$-\frac{2}{3^n} \operatorname{Re} \left(\sum_{r \in \mathbb{Z}_3^n} \hat{f}(r) E(a^r) \right) \leq 1 \quad \text{for each } f \in \mathcal{F}_{3,n}. \tag{10}$$

8. Relations with known inequalities

There had been some attempts to generalize CHSH inequalities to dimensions greater than 2. We show here that, at least for the trichotomic case, the CHSH inequality for qutrits described in [6] and the CGLMP inequality presented in [8] are easily obtained by combination of two homogeneous Bell inequalities.

CHSH for qutrits

The authors of [6] formulate an inequality named the CHSH inequality for qutrits. This inequality involves expected values and is equivalent to the one found in [17] which was formulated using joint probabilities. With our notations, the CHSH inequality for qutrits reads

$$S \leq 2 \quad (11)$$

where

$$S = \operatorname{Re} (E(a_1 a_2) + E(a_1 b_2) - E(b_1 a_2) + E(b_1 b_2)) + \frac{1}{\sqrt{3}} \operatorname{Im} (E(a_1 a_2) - E(a_1 b_2) - E(b_1 a_2) + E(b_1 b_2)).$$

We show how this inequality is strongly related to the ones we found. First, we remark that

$$\begin{aligned} S &= \frac{2}{\sqrt{3}} \operatorname{Im} (-\omega^2 E(a_1 a_2) + \omega E(a_1 b_2) + \omega^2 E(b_1 a_2) - \omega^2 E(b_1 b_2)) \\ &= \frac{2}{\sqrt{3}} \operatorname{Re} (i\omega^2 E(a_1 a_2) - i\omega E(a_1 b_2) - i\omega^2 E(b_1 a_2) + i\omega^2 E(b_1 b_2)) \\ &= \frac{2}{3} \operatorname{Re} ((1 - \omega)E(a_1 a_2) + (1 - \omega^2)E(a_1 b_2) + (\omega - 1)E(b_1 a_2) + (1 - \omega)E(b_1 b_2)) \end{aligned}$$

because $i\omega^2 = (1 - \omega)/\sqrt{3}$ and $i\omega = (\omega^2 - 1)/\sqrt{3}$. Then, observe that $a_1^2 a_2^2 = (a_1 a_2)^*$. Hence we can take conjugates to obtain

$$S = \frac{2}{3} \operatorname{Re} ((1 - \omega^2)E(a_1^2 a_2^2) + (1 - \omega)E(a_1^2 b_2^2) + (\omega^2 - 1)E(b_1^2 a_2^2) + (1 - \omega^2)E(b_1^2 b_2^2)) = -\frac{2}{9} \operatorname{Re} T$$

where

$$T = 3((\omega^2 - 1)E(a_1^2 a_2^2) + (\omega - 1)E(a_1^2 b_2^2) + (1 - \omega^2)E(b_1^2 a_2^2) + (\omega^2 - 1)E(b_1^2 b_2^2)). \quad (12)$$

Now we can express T using homogeneous Bell polynomials. For $d = 3$, our homogeneous Bell polynomials are formed with the 9 monomials

$$A_1^2 A_2^2, \quad A_1^2 A_2 B_2, \quad A_1^2 B_2^2, \quad A_1 B_1 A_2^2, \quad A_1 B_1 A_2 B_2, \quad A_1 B_1 B_2^2, \quad B_1^2 A_2^2, \quad B_1^2 A_2 B_2, \quad B_1^2 B_2^2.$$

Among these monomials, 4 of them involve only one observable for each party. The only homogeneous Bell polynomials which involve only these 4 monomials are trivial ones (e.g. $9A_1^2 A_2^2$). However, there are sums of **two** homogeneous Bell polynomials for which the terms involving incompatibles measurements cancel each other. One of these is obtained with the maps f and g from \mathbb{Z}_3^2 to \mathcal{U} given by the following vectors of values:

$$f : (\omega, 1, 1, \omega, \omega, \omega, \omega, \omega^2, 1) \quad \text{and} \quad g : (\omega^2, \omega^2, \omega, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2).$$

Their Fourier transforms are given by:

$$\begin{aligned} \hat{f} &: (4\omega + 2, \omega - 1, 4\omega - 1, -2\omega - 1, \omega - 1, \omega + 2, \omega + 5, \omega - 1, -2\omega - 4), \\ \hat{g} &: (-7\omega - 8, -\omega + 1, -\omega - 2, 2\omega + 1, -\omega + 1, -\omega - 2, 2\omega + 1, -\omega + 1, -\omega - 2) \end{aligned}$$

so their sum has values vector

$$\hat{f} + \hat{g} : (-3\omega - 6, 0, 3\omega - 3, 0, 0, 0, 3\omega + 6, 0, -3\omega - 6).$$

8. Relations with known inequalities

We obtain the polynomial

$$\begin{aligned}\mathcal{P}_f + \mathcal{P}_g &= (-3\omega - 6)A_1^2A_2^2 + (3\omega - 3)A_1^2B_2^2 + (3\omega + 6)B_1^2A_2^2 - (3\omega + 6)B_1^2B_2^2 \\ &= 3((\omega^2 - 1)A_1^2A_2^2 + (\omega - 1)A_1^2B_2^2 + (1 - \omega^2)B_1^2A_2^2 + (\omega^2 - 1)B_1^2B_2^2).\end{aligned}\quad (13)$$

Using Inequality (10) for both polynomials, we get

$$-\frac{2}{9} \operatorname{Re} \sum_{r \in \mathbb{Z}_3^2} (\hat{f}(r) + \hat{g}(r)) E(a^r) \leq 2, \quad (14)$$

or $-\frac{2}{9} \operatorname{Re} T \leq 2$. This is exactly Inequality (11).

Of course, there are many other pairs of homogeneous Bell polynomials such that the terms involving non compatible observables cancel each. For example, with the maps

$$f : (1, 1, 1, 1, 1, 1, \omega, 1, 1) \quad \text{and} \quad g : (\omega^2, 1, \omega, \omega, \omega, \omega, \omega^2, \omega^2, \omega),$$

which have Fourier transforms

$$\begin{aligned}\hat{f} &: (\omega + 8, \omega - 1, \omega - 1, \omega + 2, \omega + 2, \omega + 2, -2\omega - 1, -2\omega - 1, -2\omega - 1), \\ \hat{g} &: (2\omega - 2, -\omega + 1, -4\omega - 5, -\omega - 2, -\omega - 2, -\omega - 2, -\omega + 4, 2\omega + 1, -4\omega - 2),\end{aligned}$$

the sum of the two corresponding Homogeneous Bell polynomials is

$$\begin{aligned}\mathcal{P}_f + \mathcal{P}_g &= 3((1 - \omega^2)A_1^2A_2^2 + (\omega^2 - 1)A_1^2B_2^2 + (1 - \omega)B_1^2A_2^2 + (\omega^2 - \omega)B_1^2B_2^2) \\ &= -\frac{9}{2} \left((-A_1^2A_2^2 + A_1^2B_2^2 - B_1^2A_2^2) + \frac{i}{\sqrt{3}}(-A_1^2A_2^2 + A_1^2B_2^2 + B_1^2A_2^2 + 2B_1^2B_2^2) \right).\end{aligned}$$

Hence, we obtain the new inequality

$$\operatorname{Re} (-E(a_1a_2) + E(a_1b_2) - E(b_1a_2)) + \frac{1}{\sqrt{3}} \operatorname{Im} (-E(a_1a_2) + E(a_1b_2) + E(b_1a_2) + 2E(b_1b_2)) \leq 2.$$

In fact, this inequality can be obtained from (11) using the symmetries discussed in Section 5 (a multiplication by a global phase ω is responsible for obtaining the unusual term $\operatorname{Im} 2E(b_1b_2)$). An exhaustive search, among the sums of two trichotomic homogeneous Bell inequalities without terms involving incompatible observables, gave no inequality which cannot be obtained from (11) using symmetries.

CGLMP

The authors of [8] consider the case of two parties, two observables per parties, and general d outcomes. They introduce the following expression, using their original notations:

$$\begin{aligned}&(p(A_1 = B_1 + k) + p(B_1 = A_2 + k + 1) + p(A_2 = B_2 + k) + p(B_2 = A_1 + k)) \\ &- (p(A_1 = B_1 - k - 1) + p(B_1 = A_2 - k) + p(A_2 = B_2 - k - 1) + p(B_2 = A_1 - k - 1)).\end{aligned}$$

We rewrite it in accordance with the conventions used in the present paper (we use \mathcal{U} valued observables, with multiplicative notation ; also we use letters to denote different observables and use indices to refer to parties, while opposite convention was used in [8]). We also change two of the observables to their conjugates. The rewritten expression, obtained with the substitutions $A_1 \rightarrow a_1, B_1 \rightarrow a_2^*, A_2 \rightarrow b_1, B_2 \rightarrow b_2^*$, reads

$$\begin{aligned}S_k &:= (p(a_1 = \omega^k a_2^*) + p(a_2^* = \omega^{k+1} b_1) + p(b_1 = \omega^k b_2^*) + p(b_2^* = \omega^k a_1)) \\ &- (p(a_1 = \omega^{-k-1} a_2^*) + p(a_2^* = \omega^{-k} b_1) + p(b_1 = \omega^{-k-1} b_2^*) + p(b_2^* = \omega^{-k-1} a_1)).\end{aligned}$$

For $d = 3$, the CGLMP inequality has the simple form $S_0 \leq 2$.

9. Violations by Quantum Mechanics

To compare with homogeneous Bell inequalities, we have to find a formulation of S_k involving expected values instead of probabilities. We use

$$\omega^{-u}E(xy) = \sum_{t=0}^{d-1} \omega^{t-u} p(xy = \omega^t) = \sum_{t=0}^{d-1} \omega^{t-u} p(y = \omega^t x^*) = \sum_{t=0}^{d-1} \omega^t p(y = \omega^{t+u} x^*).$$

In the particular case $d = 3$, we obtain

$$\operatorname{Re}(\omega^{-u}E(xy)) = p(y = \omega^u x^*) - \frac{1}{2}(1 - p(y = \omega^u x^*)) = \frac{3}{2}p(y = \omega^u x^*) - \frac{1}{2}.$$

Hence, $p(y = \omega^u x^*) = \frac{2}{3} \operatorname{Re}(\omega^{-u}E(xy)) + \frac{1}{3}$. From this, we get

$$S_k = \frac{2}{3} \operatorname{Re}((\omega^{-k} - \omega^{k+1})E(a_2 a_1) + (\omega^{k+1} - \omega^{-k})E(b_1 a_2) + (\omega^{-k} - \omega^{k+1})E(b_2 b_1) + (\omega^k - \omega^{-k-1})E(a_1 b_2))$$

and, as a special case,

$$\begin{aligned} S_0 &= -\frac{2}{3} \operatorname{Re}((\omega - 1)E(a_1 a_2) + (1 - \omega)E(b_1 a_2) + (\omega^2 - 1)E(a_1 b_2) + (\omega - 1)E(b_1 b_2)) \\ &= -\frac{2}{9} \operatorname{Re} 3((\omega^2 - 1)E(a_1^2 a_2^2) + (1 - \omega^2)E(b_1^2 a_2^2) + (\omega - 1)E(a_1^2 b_2^2) + (\omega^2 - 1)E(b_1^2 b_2^2)). \\ &= -\frac{2}{9} \operatorname{Re} T \quad \text{with the same } T \text{ as in (12)}. \end{aligned}$$

Hence, for $d = 3$, the CGLMP inequality is equivalent to Inequality (11) and can be obtained from the same two homogeneous Bell inequalities. We have not yet been able to clarify links between CGLMP and homogeneous inequalities for $d > 3$. It seems more difficult in this case to conciliate the joint probability approach of [8] with the correlation functions approach we follow.

9. Violations by Quantum Mechanics

At this point, we have only considered local-realistic models. The polytope Ω we have made explicit using homogeneous Bell inequalities is the domain accessible with such models. However, the primary aim of Bell inequalities was (at least historically) to compare local-realistic theories with Quantum Mechanics. The main success of the original and CHSH Bell inequalities, was due to the fact that Quantum Mechanics violate them, hence they provided the proof that quantum indeterminacy cannot be explained by hidden variables. We now show that Quantum Mechanics also violates homogeneous Bell inequalities, and that this fact could be, in principle, checked by experiment.

There exists a difficulty in our setting, which did not appear in the $d = 2$ case. Multidimensional homogeneous Bell polynomials involve products of variables, some of them corresponding to observables of the same party. In Quantum Mechanics, such observables corresponds to non commuting operators and their values cannot be simultaneously obtained, and this prevents to observe violations this way. However, there are important cases where such products of observables are themselves observables. This is our key tool now.

Generalized Pauli matrices

We use the following multidimensional generalization (found for example in [28] and [14]) of Pauli (or spin) matrices. Let

$$X = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix} = \sum_{i=0}^{d-1} |i+1 \bmod d\rangle \langle i| \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{d-1} \end{pmatrix} = \sum_{i=0}^{d-1} \omega^i |i\rangle \langle i|,$$

where the kets $|0\rangle, \dots, |d-1\rangle$ form an orthonormal basis of \mathbb{C}^d (in fact, the eigenbasis of Z). The matrices X and Z have order d and satisfy $ZX = \omega XZ$. The *generalized Pauli matrices* are the following $d+1$ unitary matrices:

$$Z, X, XZ, \dots, XZ^{d-1}. \tag{15}$$

The following two results are easy to show. The first one is about eigenvalues as these are the possible outcomes of measurements in Quantum Mechanics.

9. Violations by Quantum Mechanics

9.1. — Proposition. *Let k be an integer. The eigenvalues of XZ^k are the ω^j (with $0 \leq j \leq d-1$) when d is odd or when k is even. They are the $\rho\omega^j$, with $\rho = \exp(i\pi/d)$, when d is even and k is odd.*

PROOF — By expanding the characteristic polynomial of XZ^k along the last column, we obtain :

$$\begin{aligned} \det(XZ^k - \lambda I) &= (-1)^{d-1} \omega^{k(d-1)} \omega^{k(0+1+\dots+(d-2))} - \lambda(-\lambda^{d-1}) \\ &= (-\lambda)^d - (-1)^d \omega^{k(d-1)+k(d-1)(d-2)/2} = (-\lambda)^d - (-1)^d \omega^{k(d-1)d/2} \\ &= \begin{cases} -\lambda^d + 1^{k(d-1)/2} = 1 - \lambda^d & \text{when } d \text{ is odd,} \\ \lambda^d - (-1)^{k(d-1)} = \lambda^d - (-1)^k & \text{when } d \text{ is even.} \end{cases} \end{aligned}$$

Hence the eigenvalues are the solutions of equation $\lambda^d = 1$ when d is odd or k is even ; and of $\lambda^d = -1$ when d is even and k is odd. \square

9.2. — Lemma. *For any integers k, e , the following relation holds*

$$(XZ^k)^e = \omega^{ke(e-1)/2} X^e Z^{ke}.$$

PROOF — We leave it to the reader. It can be done by induction over e , using $ZX = \omega XZ$. \square

Unitary observables

It is shown in [3], and also in [25], that when d is a power of a prime, the bases consisting of the normalized eigenvectors of the $d+1$ Pauli matrices given by (15) form $d+1$ *Mutually Unbiased Bases* [15]. Paterek [22] explains that these generalized Pauli matrices can be used as unitary observables, instead of more classical Hermitian operators. (Note that for the usual case $d=2$, the matrices X and Z are Hermitian as well as unitary.) These are clearly the operators we need, as we considered complex valued observables.

To determine quantum violations of homogeneous Bell inequalities, we have to evaluate expected values of operators of the form $X^{d-1-r} Z^r$, for $0 \leq r \leq d-1$. These are unitary observables, and it should be possible to directly obtain outcomes of them (without measuring outcomes of X and Z). At least, we can rely on the better known generalized Pauli operators using the following Proposition.

9.3. — Proposition. *Let $0 \leq r \leq d-1$ and assume d prime. It is possible to experimentally obtain values for $X^{d-1-r} Z^r$ in order to compute the corresponding expected value.*

PROOF — When $r = d-1$, just make a measurement with operator Z on each sample, and raise the outcomes to power $d-1$. We now assume that $0 \leq r \leq d-2$. Thus $1 \leq d-1-r \leq d-1$. As d is prime, then $(d-1-r)$ is invertible modulo d and it is possible to find an integer k such that $1 \leq k \leq d-1$ and $k(d-1-r) \equiv r$ modulo d . From Lemma 9.2, we get

$$(XZ^k)^{d-1-r} = \omega^{k(d-1-r)(d-2-r)/2} X^{d-1-r} Z^{k(d-1-r)} = \omega^{k(r+1)(r+2)/2} X^{d-1-r} Z^r.$$

Hence, we have to make a measurement with operator XZ^k on each sample, raise the outcomes to the power $d-1-r$, and multiply the results with $\omega^{-k(r+1)(r+2)/2}$. \square

Now, we are able to compute some violations of homogeneous Bell inequalities by Quantum Mechanics, with the quantum operators X and Z in place of the classical operators A_i and B_i respectively. Hence, we consider the following quantum counterparts of our homogeneous Bell polynomials (7):

$$Q_f = \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) A_{QM}^r \quad \text{where } A_{QM}^r := \bigotimes_{i=1}^n (X^{d-1-r_i} Z^{r_i}).$$

A quantum state $|\phi\rangle$ will violate the corresponding homogeneous Bell inequality if the condition

$$\operatorname{Re} \left(\frac{\rho}{d^n \cos(\pi/d)} \langle \psi | Q_f | \psi \rangle \right) \leq 1$$

is **not** satisfied.

10. Conclusion

Maximal violations

We end this paper on a remark about maximal violations of inequalities for qutrits. A known fact about the CHSH for qutrits and CGLMP inequalities is that the maximal violation is not obtained with a maximally entangled state. The fact that they can be written as the sum of two homogeneous inequalities (but each involving incompatible observables) may provide an explanation of this surprising feature.

Maximal violations of these inequalities were studied in [2] and [12]. Their authors used quantum operators corresponding to tritter measurements $A_j = U_j^\dagger Z U_j$ and $B_j = V_j^\dagger Z V_j$ where

$$U_j = \frac{1}{\sqrt{3}} H \left(\sum_{t=0}^{d-1} \exp(i\phi_j(t)) |t\rangle\langle t| \right) \quad \text{and} \quad V_j = \frac{1}{\sqrt{3}} H \left(\sum_{t=0}^{d-1} \exp(i\varphi_j(t)) |t\rangle\langle t| \right)$$

and ϕ_j and φ_j are triples of phases. For the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle),$$

the maximal violation of (11) is known to be $\frac{2}{9}(6 + 4\sqrt{3}) \simeq 2.8729$ and is obtained with the phases $\phi_1 = (0, 0, 0)$, $\varphi_1 = (0, \pi/3, 2\pi/3)$, $\phi_2 = (0, \pi/6, 2\pi/6)$ and $\varphi_2 = (0, -\pi/6, -2\pi/6)$. This violation can be computed by the formula

$$-\frac{2}{9} \operatorname{Re} \langle \psi | \mathcal{P}_f + \mathcal{P}_g | \psi \rangle \quad (16)$$

after substitution of the tritter operators to the A_j and B_j variables.

We computed separately the violations due to the term involving \mathcal{P}_f and to the term involving \mathcal{P}_g . We found respectively $\simeq 1.6931$ and $\simeq 1.1799$. We did also the same computation with the state

$$\frac{1}{\sqrt{2+\gamma^2}} (|00\rangle + \gamma|11\rangle + |22\rangle) \quad \text{with } \gamma = \frac{\sqrt{11}-\sqrt{3}}{2}$$

which was found in [12] to achieve better violation, while not being maximally entangled. The two terms involved in (16) we obtained are respectively $\simeq 1.6895$ and $\simeq 1.2253$, in accordance with the total violation which was known to be $1 + \sqrt{11/3} \simeq 2.9149$. We observe that the violation of the term involving f is not maximal when the sum is maximal. The question whether each term is maximally violated by a maximally entangled state remains open.

10. Conclusion

In this paper, we defined homogeneous Bell inequalities and we showed that they correspond to the boundaries of the domain accessible with local-realistic models, in a correlation functions setting with roots of unity as outcomes, for the general multipartite and multidimensional case with two observables per party. We studied homogeneous Bell polynomials and their symmetries. It turns out that the classical domain is the image under DFT of a polytope obtained from the canonical basis, and we used this fact to compute its dual. With this, we were able to show that the homogeneous Bell inequalities form a complete set.

This complete set defines a d^{d^n} facets polytope, belonging to a complex vector space of dimension d^n . In the particular case $d = 2$, this complete set reduces to the one obtained in [31] and [32], which defines a 2^{2^n} facets polytope in a real (because the convex hull of $\{\pm 1\}$ is included in \mathbb{R}) vector space of dimension 2^n .

We explored the case $d = 3$, in particular for the two parties case. We also showed that the CHSH inequality for qutrits and the CGLMP inequality for $d = 3$ are obtained from ours.

Then we considered violations by Quantum Mechanics, using the observables provided by generalized Pauli matrices. We explained how violations of homogeneous Bell inequalities could in principle be observed. We computed some violations in the trichotomic case in relation with the maximal violation of known inequalities.

The complex valued correlation function we used is a natural mathematical generalization of the two-dimensional one. Fu in [11] argued that it has also a physical meaning, at least in Quantum Mechanics. It was a crucial and fruitful ingredient in the present work, and this raises interrogations about the precise extent of this physical meaning. Also, complex valued observables provided by the generalized Pauli matrices were a key tool for computing violations by removing incompatible observables.

11. Bibliography

- [1] A. ACÍN, J.L. CHEN, N. GISIN, D. KASZLIKOWSKI, L.C. KWEK, C.H. OH, M. ŻUKOWSKI : *Coincidence Bell inequality for three three-dimensional systems*. Physical Review Letters 92, 250404 (2004).
- [2] A. ACÍN, T. DURT, N. GISIN, J.L. LATORRE : *Quantum non-locality in two three level systems*. Physical Review A, 65, 052325 (2002).
- [3] S. BANDYOPADHYAY, P.O. BOYKIN, V. ROYCHOWDHURY, F. VATAN : *A new proof for the existence of mutually unbiased bases*. Algorithmica 34, 512-528 (2002).
- [4] J.S. BELL : *On the Einstein Podolsky Rosen paradox*. Physics 1, 195 (1964).
- [5] W.L. BRIGGS : *The DFT: an owners' manual for the Discrete Fourier Transform*. Society for Industrial Mathematics, 1987.
- [6] J.-L. CHEN, D. KASZLIKOWSKI, L.C. KWEK, C.H. OH : *Wringing out new Bell inequalities for three-dimensional systems (qutrits)*. Modern Physics Letters A 17, 2231 (2002).
- [7] J.F. CLAUSER, M.A. HORNE, A. SHIMONY, R.A. HOLT : *Proposed experiment to test local hidden variables theories*. Physical Review Letters 23, 880 (1969).
- [8] D. COLLINS, N. GISIN, N. LINDEN, S. MASSAR, S. POPESCU : *Bell inequalities for arbitrarily high-dimensional systems*. Physical Review Letters 88, 040404 (2002).
- [9] A. FINE : *Hidden variables, joint probabilities, and the Bell inequalities*. Physical Review Letters 48, 291 (1982).
- [10] M. FROISSART : *Constructive Generalization of Bell's inequalities*. Nuovo Cimento Della Società Italiana Di Fisica 64B(2), 241–251 (1981).
- [11] L.-B. FU : *General correlation functions of the Clauser-Horne-Shimony-Holt inequality for arbitrarily high-dimensional systems*. Physical Review Letters 92, 130404 (2004).
- [12] L.-B. FU, J.-L. CHEN, X.-G. ZHAO : *Maximal violation of the Clauser-Horne-Shimony-Holt inequality for two qutrits*. Physical Review A 68, 022323 (2003).
- [13] A. GARG, N.D. MERMIN : *Farkas lemma and the nature of reality: statistical implications of quantum correlations*. Foundations of Physics 14(1), 1 (1984).
- [14] D. GOTTESMAN : *Fault-tolerant quantum computation with higher-dimensional systems*. Chaos, Solitons & Fractals, vol. 10, n. 10, 1749-1758 (1999).
- [15] I.D. IVANOVIC : *Geometrical Description of quantum state determination*. Journal of Physics A, 14, n. 12, 3241–3245.
- [16] D. KASZLIKOWSKI, P. GNACIŃSKI, M. ŻUKOWSKI, W. MIKLASZEWSKI, A. ZEILINGER : *Violations of local realism by two entangled N -dimensional systems are stronger than for two qubits*. Physical Review Letters 85, 4418 (2000).
- [17] D. KASZLIKOWSKI, L.C. KWEK, J.L. CHEN, M. ŻUKOWSKI, C.H. OH : *Clauser-Horne inequality for three-state systems*. Physical Review A 65, 032118 (2002).
- [18] D. KASZLIKOWSKI, D.K.L. OI, M. CHRISTANDL, K. CHANG, A. EKERT, L.C. KWEK, C.H. OH : *Quantum cryptography based on qutrit Bell inequalities*. Physical Review A 67, 012310 (2003).
- [19] E.R. LOUBENETS : *On the probabilistic description of a multipartite correlation scenario with arbitrary numbers of settings and outcomes*. Journal of Physics A: Mathematical and Theoretical 41, 445303 (2008).
- [20] E.R. LOUBENETS : *Multipartite Bell-type inequalities for arbitrary numbers of settings and outcomes per site*. Journal of Physics A: Mathematical and Theoretical 41, 445304 (2008).

11. Bibliography

- [21] LL. MASANES : *Tight Bell inequality for d -outcome measurements correlations*. Quantum Information & Computation 3(4), 345 (2003).
- [22] T. PATEREK : *Measurements on composite qudits*. Physics Letters A, 367, 57-64 (2007).
- [23] A. PERES : *All the Bell inequalities*. Foundations of Physics 29, 589-614 (1999).
- [24] I. PITOWSKY, K. SVOZIL : *Optimal tests for quantum non locality*. Physical Review A 64, 014102 (2001).
- [25] A.O. PITTENGER, M.H. RUBIN : *Mutually unbiased bases, generalized spin matrices and separability*. Linear Algebra and its Applications 390, 255-278 (2004).
- [26] G. SCHACHNER : *The structure of Bell inequalities*. arXiv:quant-ph/0312117 (2003).
- [27] H.H. SCHAEFER : *Topological Vector Spaces*. Springer, Berlin, 1980.
- [28] J. SCHWINGER : *Quantum Mechanics — Symbolism of Atomic Measurements*. edited by B.-G. Englert, Springer, Berlin, 2001.
- [29] E. SHCHUKIN : *Bell inequalities, classical cryptography and fractals*. arXiv:quant-ph/0703259v2 (2007).
- [30] W. SON, J. LEE, M.S. KIM : *Generic Bell inequalities for multipartite arbitrary dimensional systems*. Physical Review Letters 96, 060406 (2006).
- [31] R.F. WERNER, M.M. WOLF : *All-multipartite Bell-correlation inequalities for two dichotomic observables per site*. Physical Review A, 64, 032112 (2001).
- [32] M. ŻUKOWSKI, Č. BRUKNER : *Bell's theorem for general N -qubit states*. Physical Review Letters 88, 210401 (2002).